Understanding the ergodic hypothesis via analogies

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The Ergodic Hypothesis is a hypothesis in Statistical Mechanics that relates the microscopic motion of particles with the macroscopic average, i.e., the observed property. Despite its importance, didactically its understanding is not easy due to technical issues. Therefore, in this article we propose analogies in order to clarify some important features of the referred hypothesis. Our starting point is the perception that the same macroscopic property, i.e., the average of the movement, can be calculated by different procedures. After that, we build the same average in a more convenient way. We do not have as objectives to contemplate advanced implications of the referred hypothesis. On such cases, some papers will be referenced.

I. INTRODUCTION

In many theoretical contexts, technical sophistication often blurs the meaning and intuition behind a theory. Besides, the difficulties that enclosure fundamental questions usually have an inhibitor effect on curious eyes. Accepting such truth, textbooks’ authors follow the tendencies of technical tutorials, banning from their pages crucial explanatory steps, which may possibly impair the proper discernment of a hypothesis. With that concern in mind, we propose some analogies to help explaining an important topic in Statistical Mechanics: the Ergodic Hypothesis (EH).

A. Statistical Mechanics

In the last decades of 19th century, great advances were made, concerning the comprehension of matter structure. Despite the resistance of the energetics’ group 1, the works by Maxwell, Boltzmann, Gibbs and others were successful on interpreting matter as composed by the minors entities called atoms. Under this new perspective, many obstacles needed to be overcome, and EH appears with singular voice to solve some problems.

Briefly, the EH states that the temporal average of the movement of molecules and atoms is equal to the spatial average (in phase space, see eqs. (1) and (9)). In other words, this means that a macroscopic property can be interpreted as an average taken over different data.

Despite the fundamental nature of the hypothesis, its presentation in many textbooks is underestimated, giving an impression of being either a mere step in mathematical calculations or even a self-evident conclusion. We mention some examples: ref. [3] just mentions it. Refs. [4, 5] make some comments, specially ref. [6] whose dense book about fundamentals of Statistical Mechanics has discussions about EH’s implications. However, no one emphasizes its intuitive meaning. Other texts about Kinetic Theory of Gases [7, 8] make use of that meaning without justifying or even commenting about it. Ref. [9] is an exception that introduces EH to connect micro- and macroscopic scales. In short, even on those texts that recognizes EH’s fundamental role, its hard to comprehend the reasons that take us to adopt it.

We mention refs. [10–12] as good sources of information, despite the fact they are high technical level readings. We may still mention ref. [13] as an example that shows the actual relevance of this subject, which deals with the Quantum Ergodic Theorem according to John von Neumann.

II. DESCRIPTIONS OF THE MICROSCOPIC STATE

The description of a physical system, such as an ideal gas with N particles, is usually carried out in terms of 3N spatial coordinates q1,...,q3N ≡ q; and the respective 3N conjugated momenta p1,...,p3N ≡ p. We are then considering the Hamiltonian formalism. We still define a differential volume of phase space by dω ≡ dq1…dq3N dp1…dp3N.

Imagine that we intend to calculate an (intensive) macroscopic property Gobs, where obs means observable, taking the atomics hypothesis as true. An usual way of trying to extract a simple and unique value from an infinite number of irregularly moving particles is to properly conceive an average of the motion (position or velocity) of the particles.

With the aim of testing the validity of the previous procedure, we will perform an experiment. Due to the fact that any experimental measurement occurs in a finite time τ, the required average must be evaluated during that interval2 [9].

1 The energetics were those who disagreed that the matter would be subdivided into smaller entities, the atoms. As representatives of this current of thought include Helm and Ostwald [1, 2].

2 Implicitly, we are considering that the system reaches equilibrium during the finite time interval τ, otherwise there would be no use in computing this average. So we focus our attention on studying systems in equilibrium [14].
Mathematically, this is translated as,
\[ \mathcal{G}_{\text{obs}} = \bar{G}(\tau) = \frac{1}{\tau} \int_{0}^{\tau} G(q, p; t) \, dt . \tag{1} \]

Even if the function \( G \) is known, it is impracticable to compute all \( 6N \) equations of movement together with their precise initial conditions. In this way, the temporal average is useless at getting some information.

How could we obtain \( \mathcal{G}_{\text{obs}} \)? Is eq. (1) adequate? Is that the only possible way to calculate the macroscopic measurement?

III. THE CONSUMER AND ITS EXPENDITURES

Imagine that we wish to know the mean daily consumption (macroscopic property \( \mathcal{Q}_{\text{obs}} \)) of one consumer; we call him \( \ell^* \). In this case, we propose to track him during a period of a month (30 days), after which, we calculate the average. In mathematical notation,
\[ \mathcal{Q}_{\text{obs}} = \bar{Q}(d) = \frac{1}{30} \sum_{d=1}^{30} Q(d) . \tag{2} \]

We can now suppose that we do not have an entire month to observe \( \ell^* \); in fact, we have just one day. How could we know that average? Would it be possible to know \( \mathcal{Q}_{\text{obs}} \) so far?

If we believe – here is the thesis – that there are many consumers \( L \) with similar profiles of \( \ell^* \), i.e., a socioeconomic group \( \mathcal{L} \) to which \( \ell^* \) belongs, we could observe many of them for just one day, taking notes of their expenditures and calculating the desired average:
\[ \langle Q(\ell) \rangle = \frac{1}{L} \sum_{\ell=1}^{L} Q(\ell) . \tag{3} \]

The family of \( L \) consumers sent to shopping is commonly called “ensemble”. If we take \( \ell^* \) as one of the many possibilities of the ensemble \( \mathcal{L} \), we say that \( \ell^* \) is a realization of ensemble [10, 15, 16].

It is obvious that in a certain day the consumer \( \ell^* \) can perform and probably he will, different expenditures from that performed by the consumer \( i \), which will be different from those of \( j \), and so on.

In fact, what we are supposing is that the unique consumer \( \ell^* \) could perform, in any of the 30 days of the month, the several expenditures of his pairs, in such way that the average of the socioeconomic class \( \mathcal{L} \) taken during one single day equals the average of the unique consumer \( \ell^* \) taken during a whole month.

Hypothesis:

\[ \begin{pmatrix} \text{L consumers} \\ \text{in just one day} \end{pmatrix} = \begin{pmatrix} \text{an unique consumer} \\ \text{\ell^* during a whole month} \end{pmatrix} . \]

In this way, we change a temporal average \( \bar{Q}(d) \) for another one built on socioeconomic class \( \mathcal{L} \), \( \langle Q(\ell) \rangle \). The important fact is that we consider ourselves successful in satisfying our curiosity about the mean daily consumption of the specific consumer \( \ell^* \). Mathematically,
\[ \mathcal{Q}_{\text{obs}} = \bar{Q}(d) = \langle Q(\ell) \rangle . \tag{4} \]

IV. EXPANDING THE ANALOGY

The idea above shows our goal: exchanging the average based on the time evolution of a single agent, for another based on a set of similar agents.

With this motivation we choose not to hold ourselves in the class \( \mathcal{L} \). Instead of sending the consumers of \( \mathcal{L} \) to shopping in just one day, we can – in an abstract way – give life to each possible expenditures instantaneously, i.e., in a time interval \( \Delta t \to 0 \). This means we are changing the focus from the class of consumers \( \mathcal{L} \) to the expenditures space \( \mathfrak{M} \) and eliminating the time dependence of the average.

We know that there are an infinity of possibilities of expenditures (from one simple candy from the neighborhood grocery store to a luxurious car)\(^3\), then the sum must be replaced by an integral. Besides, it is natural to believe that buying a simple candy is much more common than buying a sport car. So we need to assign a probability density \( f(m) \) to the occurrence of each expenditure \( m \) of the set \( \mathfrak{M} \).

In mathematical terms, we now express the mean daily consumption,
\[ \langle Q(m) \rangle = \int_{\mathfrak{M}} f(m) Q(m) \, dm \tag{5} \]

and,
\[ \int_{\mathfrak{M}} f(m) \, dm = 1 \tag{6} \]

since \( f \) is a probability density.

This procedure allows us to know the same property \( \mathcal{Q}_{\text{obs}} \) in three different ways,
\[ \mathcal{Q}_{\text{obs}} = \bar{Q}(d) = \langle Q(\ell) \rangle_{\mathcal{L}} = \langle Q(m) \rangle_{\mathfrak{M}} . \tag{7} \]

V. THE ERGODIC HYPOTHESIS

The EH consists in changing the temporal average of eq. (1) for a spatial average. More precisely, for an average on the phase space of Hamiltonian formalism.

In this formalism, we express the microscopic state of a system with just one point in a space of \( 6N \) dimensions. Then instead of tracking a long path of this single point during the time interval \( \tau \) and computing the integral in eq. (1), we will imagine an ensemble of similar points\(^4\) and assign to each portion of that ensemble a probability density \( f_N(q, p; t) \) that quantifies the movements in all directions of phase space. Afterwards, we integrate the possibilities in all directions to obtain the macroscopic tendency.

\[ \mathcal{G}_{\text{obs}} = \langle G(t) \rangle = \int_{\Omega} f_N(q, p; \Delta t \to 0) G(q, p; \Delta t \to 0) \, d\omega \tag{8} \]

\(^3\) We should have in mind that the only possible expenditures are those that characterize the socioeconomic group \( \mathcal{L} \).

\(^4\) The similarity lies in the fact that all the replicas of the point must correspond to the same macroscopic state.
where \( d\omega \) is the volume element of a phase space region \( \Omega \). We may even drop the explicit time dependence and write it down,

\[
G_{\text{obs}} = \langle G(t) \rangle = \int_{\Omega} f_N(q,p)G(q,p)d\omega. \tag{9}
\]

A. Why choose an average over the phase space?

Think of a landscape with mountains, valleys, cliffs, plains, rivers, sea, etc. Imagine now that you are with a group of friends in some point of that landscape, on a valley, for example. You are lost and tired, and a question arise: where should I go? Two main alternatives appears:

i. if everybody walk together as an unique point during many days in certain direction, after that long period the group will ponder the displacement, i.e., compute an average.

ii. or, individually, each one chooses a direction to go, and after some hours, return with some information. Taking this informations as basis, the group will decide which direction to follow.

In the first alternative, the group goes as unity. During the displacement, it ponders about the mountains, rivers, cliffs (in other words, the topography and geometry) of the landscape. Depending on amount of food and fatigue (energy), the group will choose a specific route. After some days, it may decide the trip path.

Look at the second alternative. Imagine that one person returns after two hours and says that there is a high climb towards the east, which means that direction to follow.

In the first alternative, the group goes as unity. During the displacement, it ponders about the mountains, rivers, cliffs (in other words, the topography and geometry) of the landscape. Depending on amount of food and fatigue (energy), the group will choose a specific route. After some days, it may decide the trip path.

Finally, refs. [11] and [12] may serve as a next step in addition to this paper, and may be used for discussing non-ergodic systems and modern advances of Ergodic Theory.

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